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Two-Dimensional Spline Functions and Best Approximations of Linear Functionals

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1. INTRODUCTION

Let ℓ be a linear functional which can be represented in the form

$$\ell(f) = \sum_{\nu=0}^{m-1} \int_a^b f^{(\nu)}(x)\, d\mu_\nu(x),$$

where $\mu_\nu(x)$ are functions of bounded variation. Suppose $a \leqslant x_1 < \cdots < x_p \leqslant b$ and $I_i \subset \{0,..., m-1\}$ are given and suppose we wish to approximate $\ell(f)$ by an expression of the type

$$\sum_{i=1}^{p} \sum_{j \in I_i} \sigma_{ij} f^{(j)}(x_i),$$

which is exact for polynomials of degree $m - 1$. It was shown by Schoenberg [6] and Ahlberg and Nilson [1] that the coefficients σ_{ij} for which the above approximation is best in the sense of Sard [5] can be obtained by operating with ℓ on an appropriate spline interpolation formula.

In the present paper we study the problem of obtaining best approximations to a certain class of linear functionals operating on functions of two variables. It will turn out that for the solution of this problem, spline interpolation formulas play the same important role as they do in the one-dimensional case.

In the next section a precise definition of the approximation problem is given. Section 3 is devoted to the construction of a two-dimensional spline interpolation formula. In the final section, a connection between best approximations of linear functionals and spline interpolation is established.

2. THE APPROXIMATION PROBLEM

Let $a_1 \leqslant a_2$ and $b_1 \leqslant b_2$ be real numbers and define

$$R = \{(x, y) \in E^2 \mid a_1 \leqslant x \leqslant b_1, a_2 \leqslant y \leqslant b_2\}.$$

Let $C^{mn}[R]$ denote the space of all real functions $g(x, y)$ for which

$$\frac{\partial^{i+j}}{\partial x^i \, \partial y^j} g(x, y), \qquad i = 0,..., m, \qquad j = 0,..., n$$

exist and are continuous in R.

We consider linear functionals ℓ over $C^{mn}[R]$ of the following type:

$$\begin{aligned} \ell(g) = \sum_{i=0}^{m-1} \sum_{j=0}^{n-1} \Big\{ &\int_{a_1}^{b_1} \int_{a_2}^{b_2} a^{ij}(x, y) \frac{\partial^{i+j}}{\partial x^i \, \partial y^j} g(x, y) \, dy \, dx \\ &+ \sum_{\eta=1}^{q} \int_{a_1}^{b_1} b_\eta^{ij}(x) \frac{\partial^{i+j}}{\partial x^i \, \partial y^j} g(x, y_\eta) \, dx \\ &+ \sum_{\xi=1}^{p} \int_{a_2}^{b_2} c_\xi^{ij}(y) \frac{\partial^{i+j}}{\partial x^i \, \partial y^j} g(x_\xi, y) \, dy \\ &+ \sum_{\xi=1}^{p} \sum_{\eta=1}^{q} \alpha_{\xi\eta}^{ij} \frac{\partial^{i+j}}{\partial x^i \, \partial y^j} g(x_\xi, y_\eta) \Big\}, \end{aligned} \tag{2.1}$$

where $\alpha_{\xi\eta}^{ij}$ are real constants. The functions $a^{ij}(x, y)$, $b_\eta^{ij}(x)$, and $c_\xi^{ij}(y)$ are assumed to be continuous over R and the points (x_ξ, y_η) lie in R.

For every $\nu \in \{1, 2,..., p\}$ and every $\mu \in \{1,..., q\}$, let

$$I(\nu, \mu) \subset \{(i, j) \mid i = 0,..., m-1, j = 0,..., n-1\}.$$

The possibility that certain sets $I(\nu, \mu)$ are void is not excluded.

Suppose $g \in C^{mn}[R]$ and ℓ is of type (2.1). Then we consider the problem of determining real numbers $c_{\nu\mu}^{ij}$ so that the sum on the right side of the formula

$$\ell(g) = \Big[\sum_{\nu=1}^{p} \sum_{\mu=1}^{q} \sum_{(i,j)\in I(\nu,\mu)} c_{\nu\mu}^{ij} \frac{\partial^{i+j}}{\partial x^i \, \partial y^j} g(x_\nu, y_\mu)\Big] + \mathscr{R}(g) \tag{2.2}$$

represents an approximation to $\ell(g)$ which is exact for all $g \in \Pi_{m-1,n-1}$. Here and below, $\Pi_{m-1,n-1}$ denotes the set of all real polynomials of the form

$$\sum_{i=0}^{m-1} \sum_{j=0}^{n-1} p_{ij} x^i y^j.$$

The requirement $\mathscr{R}(g) = 0$ for all $g \in \Pi_{m-1,n-1}$ is equivalent to the system of mn equations

$$\ell(x^k y^l) = \sum_{\nu=1}^{p} \sum_{\mu=1}^{q} \sum_{I(\nu,\mu)} c_{\nu\mu}^{ij} \frac{\partial^{i+j}}{\partial x^i \, \partial y^j} x^k y^l \Big|_{x=x_\nu, y=y_\mu},$$

$$k = 0,\ldots, m-1, \qquad l = 0,\ldots, n-1. \tag{2.3}$$

In case the number of available parameters $c_{\nu\mu}^{ij}$ is greater than mn, we wish to determine the $c_{\nu\mu}^{ij}$ so that (2.3) will be satisfied and the approximation will be best in the sense of Sard [5].

For this purpose, we need a two-dimensional analog of Peano's Theorem [2]. In this analog and later on we use the truncated power function $(x - \bar{x})_+^k$, defined as

$$(x - \bar{x})_+^k = \begin{cases} (x - \bar{x})^k & \text{for } x - \bar{x} \geqslant 0, \\ 0 & \text{for } x - \bar{x} < 0. \end{cases}$$

THEOREM (2.1). *Let ℓ be of type* (2.1) *and let $\ell(h) = 0$ for all $h \in \Pi_{m-1,n-1}$. Then for every $g \in C^{mn}[R]$,*

$$\ell(g) = (-1)^{m+n} \int_{a_1}^{b_1} \int_{a_2}^{b_2} K(x, y) \frac{\partial^{m+n}}{\partial x^m \, \partial y^n} g(x, y) \, dy \, dx$$

$$+ \sum_{l=0}^{n-1} (-1)^{m+l} \int_{a_1}^{b_1} K_{1l}(x) \frac{\partial^{m+l}}{\partial x^m \, \partial y^l} g(x, b_2) \, dx$$

$$+ \sum_{k=0}^{m-1} (-1)^{n+k} \int_{a_2}^{b_2} K_{2k}(y) \frac{\partial^{k+n}}{\partial x^k \, \partial y^n} g(b_1, y) \, dy,$$

where

$$K(x, y) = \frac{1}{(m-1)!\,(n-1)!} \ell_{st}((x - s)_+^{m-1} (y - t)_+^{n-1}),$$

$$K_{1l}(x) = \frac{1}{(m-1)!\,l!} \ell_{st}((x - s)_+^{m-1} (b_2 - t)^l), \qquad l = 0,\ldots, n-1,$$

$$K_{2k}(y) = \frac{1}{k!\,(n-1)!} \ell_{st}((b_1 - s)^k (y - t)_+^{n-1}), \qquad k = 0,\ldots, m-1.$$

The notation $\ell_{st}(x - s)_+^{m-1} (y - t)_+^{n-1}$ means that the functional ℓ is applied to $(s - x)_+^{m-1} (t - y)_+^{n-1}$ considered as a function of s and t.

Proof. Let $(s, t) \in R$ and $g \in C^{mn}[R]$. Then we have

$$\begin{aligned}
g(s,t) &= g(b_1, b_2) - \int_s^{b_1} \frac{\partial}{\partial x} g(x, b_2)\, dx - \int_t^{b_2} \frac{\partial}{\partial y} g(b_1, y)\, dy \\
&\quad + \int_s^{b_1} \int_t^{b_2} \frac{\partial^2}{\partial x\, \partial y} g(x, y)\, dy\, dx \\
&= g(b_1, b_2) - \int_{a_1}^{b_1} (x - s)_+^0 \frac{\partial}{\partial x} g(x, b_2)\, dx \\
&\quad - \int_{a_2}^{b_2} (y - t)_+^0 \frac{\partial}{\partial y} g(b_1, y)\, dy \\
&\quad + \int_{a_1}^{b_1} \int_{a_2}^{b_2} (x - s)_+^0 (y - t)_+^0 \frac{\partial^2}{\partial x\, \partial y} g(x, y)\, dy\, dx.
\end{aligned}$$

Integrating the last two terms $n - 1$ times by parts with respect to y, we obtain

$$\begin{aligned}
g(s,t) &= \sum_{l=0}^{n-1} (-1)^l \frac{(b_2 - t)_+^l}{l!} \frac{\partial^l}{\partial y^l} g(b_1, b_2) \\
&\quad + (-1)^n \int_{a_2}^{b_2} \frac{(y - t)_+^{n-1}}{(n-1)!} \frac{\partial^n}{\partial y^n} g(b_1, y)\, dy \\
&\quad + \sum_{l=0}^{n-1} (-1)^l \int_{a_1}^{b_1} (x - s)_+^0 \frac{(b_2 - t)_+^l}{l!} \frac{\partial^{1+l}}{\partial x\, \partial y^l} g(x, b_2)\, dx \\
&\quad + (-1)^n \int_{a_1}^{b_1} \int_{a_2}^{b_2} (x - s)_+^0 \frac{(y - t)_+^{n-1}}{(n-1)!} \frac{\partial^{1+n}}{\partial x\, \partial y^n} g(x, y)\, dy\, dx.
\end{aligned}$$

Now we integrate the last two terms $m - 1$ times by parts with respect to x. This gives

$$\begin{aligned}
g(s,t) &= \sum_{k=0}^{m-1} \sum_{l=0}^{n-1} (-1)^{k+l} \frac{(b_1 - s)_+^k}{k!} \frac{(b_2 - t)_+^l}{l!} \frac{\partial^{k+l}}{\partial x^k\, \partial y^l} g(b_1, b_2) \\
&\quad + \sum_{l=0}^{n-1} (-1)^{m+l} \int_{a_1}^{b_1} \frac{(x - s)_+^{m-1} (b_2 - t)_+^l}{(m-1)!\, l!} \frac{\partial^{m+l}}{\partial x^m\, \partial y^l} g(x, b_2)\, dx \\
&\quad + \sum_{k=0}^{m-1} (-1)^{n+k} \int_{a_2}^{b_2} \frac{(b_1 - s)_+^k (y - t)_+^{n-1}}{k!\, (n-1)!} \frac{\partial^{k+n}}{\partial x^k\, \partial y^n} g(b_1, y)\, dy \\
&\quad + (-1)^{m+n} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{(x - s)_+^{m-1} (y - t)_+^{n-1}}{(m-1)!\, (n-1)!} \frac{\partial^{m+n}}{\partial x^m\, \partial y^n} g(x, y)\, dy\, dx.
\end{aligned}$$

We apply ℓ to both sides of this expansion. Since, for $(s, t) \in R$,

$$(b_1 - s)_+^k = (b_1 - s)^k \quad \text{and} \quad (b_2 - t)_+^l = (b_2 - t)^l,$$

and since ℓ vanishes for all elements of $\Pi_{m-1,n-1}$, we obtain

$$\ell_{st}(g) = \sum_{l=0}^{n-1} (-1)^{m+l} \ell_{st} \left(\int_{a_1}^{b_1} \frac{(x-s)_+^{m-1} (b_2 - t)^l}{(m-1)!\, l!} \frac{\partial^{m+l}}{\partial x^m \partial y^l} g(x, b_2)\, dx \right)$$

$$+ \sum_{k=0}^{m-1} (-1)^{n+k} \ell_{st} \left(\int_{a_2}^{b_2} \frac{(b_1 - s)^k (y-t)_+^{n-1}}{k!\,(n-1)!} \frac{\partial^{k+n}}{\partial x^k \partial y^n} g(b_1, y)\, dy \right)$$

$$+ (-1)^{m+n} \ell_{st} \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} \frac{(x-s)_+^{m-1} (y-t)_+^{n-1}}{(m-1)!\,(n-1)!} \frac{\partial^{m+n}}{\partial x^m \partial y^n} g(x, y)\, dy\, dx \right).$$

Observe that

$$\ell_{st}((x-s)_+^{m-1} (b_2 - t)^l) \quad \text{and} \quad \ell_{st}((b_1 - s)^k (y-t)_+^{n-1})$$

are piecewise continuous functions of x and y, respectively. Furthermore,

$$\ell_{st}((x-s)_+^{m-1} (y-t)_+^{n-1})$$

is a function of x and y which is bounded over R and continuous on every set

$$R(x, y) = \{(x, y) \mid x_\nu < x < x_{\nu+1}, y_\mu < y < y_{\mu+1}\}$$
$$\nu = 0, \ldots, p, \qquad \mu = 0, \ldots, q,$$

where $x_0 = a_1$, $x_{p+1} = b_1$, $y_0 = a_2$, and $y_{p+1} = b_2$. Therefore, one can change the order of integration and application of ℓ. This completes the proof.

For later use we make here the following

Remark 1. For $\nu \in \{1, \ldots, p\}$, $\mu \in \{1, \ldots, q\}$, let

$$R^1(\nu, \mu) = \{(x, y) \in R \mid x \geqslant x_\nu\} \quad \text{and} \quad R^2(x, y) = \{(\nu, \mu) \in R \mid y \geqslant y_\mu\}.$$

Suppose $k \geqslant m$, $l \geqslant n$ and consider the functions

$$f_0(x, y) = (x - x_\nu)_+^k (y - y_\mu)_+^l, \qquad f_1(x, y) = (x - x_\nu)_+^k y^l,$$
$$f_2(x, y) = x^k (y - y_\mu)_+^l. \tag{2.4}$$

Since (in an obvious sense)

$$\begin{array}{lll} f_0(x, y) \in \Pi_{k,l} & \text{for} & (x, y) \in R^1(\nu, \mu) \cap R^2(\nu, \mu), \\ f_0(x, y) \equiv 0 & \text{for} & (x, y) \in R - R^1(\nu, \mu) \cap R^2(\nu, \mu), \\ f_1(x, y) \in \Pi_{k,l} & \text{for} & (x, y) \in R^1(\nu, \mu), \\ f_1(x, y) \equiv 0 & \text{for} & (x, y) \in R - R^1(\nu, \mu), \\ f_2(x, y) \in \Pi_{k,l} & \text{for} & (x, y) \in R^2(\nu, \mu), \\ f_2(x, y) \equiv 0 & \text{for} & (x, y) \in R - R^2(\nu, \mu), \end{array}$$

it follows that Theorem (2.1) can also be applied to functions $g(x, y)$ of type (2.4) if $k = m$ and/or $l = n$.

Because $\mathscr{R}(g)$, as defined by (2.2), is of type (2.1). It follows from Theorem (2.1) that for all $g \in C^{mn}[R]$,

$$\begin{aligned} \mathscr{R}(g) = (-1)^{m+n} \int_{a_1}^{b_1} \int_{a_2}^{b_2} K(x, y) \frac{\partial^{m+n}}{\partial x^m \, \partial y^n} g(x, y) \, dy \, dx \\ + \sum_{l=0}^{n-1} (-1)^{m+l} \int_{a_1}^{b_1} K_{1l}(x) \frac{\partial^{m+l}}{\partial x^m \, \partial y^l} g(x, b_2) \, dx \\ + \sum_{k=0}^{m-1} (-1)^{n+k} \int_{a_2}^{b_2} K_{2k}(y) \frac{\partial^{k+n}}{\partial x^k \, \partial y^n} g(b_1, y) \, dy, \end{aligned}$$

provided $\mathscr{R}(g)$ vanishes for all $g \in \Pi_{m-1,n-1}$.

Hence, the sum on the right side of (2.2) defines an approximation to $l(g)$ which is best in the sense of Sard [5] if the parameters $c_{\nu\mu}^{ij}$ are a solution of the following minimization problem: Minimize

$$\sum_{l=0}^{n-1} \int_{a_1}^{b_1} (K_{1l}(x))^2 \, dx + \sum_{k=0}^{m-1} \int_{a_2}^{b_2} (K_{2k}(y))^2 \, dy + \int_{a_1}^{b_1} \int_{a_2}^{b_2} (K(x, y))^2 \, dy \, dx \tag{2.5}$$

under the side conditions

$$\mathscr{R}(g) \equiv 0 \qquad \text{for all} \quad g \in \Pi_{m-1,n-1}, \tag{2.6}$$

where

$$\begin{aligned} K(x, y) &= \frac{1}{(m-1)!\,(n-1)!} \mathscr{R}_{st}((x - s)_+^{m-1} (y - t)_+^{n-1}), \\ K_{1l}(x) &= \frac{1}{(m-1)!\, l!} \mathscr{R}_{st}((x - s)_+^{m-1} (b_2 - t)^l), \qquad l = 0,..., n-1, \\ K_{2k}(y) &= \frac{1}{k!\,(n-1)!} \mathscr{R}_{st}((b_1 - s)^k (y - t)_+^{n-1}), \qquad k = 0,..., m-1, \end{aligned}$$

and $\mathscr{R}(g)$ is defined by (2.2).

Before we pursue this matter any further we turn to the construction of a two-dimensional spline interpolation formula, which will be a basic tool for the solution of the above minimization problem.

3. The Two-Dimensional g Spline Interpolation Formula

Let $I(\nu, \mu)$ be defined as in Section 2 and define the subsets $I_{j\nu}$ of $\{0,..., m-1\}$ and $J_{i\mu}$ of $\{0,..., n-1\}$, for $j = 0,..., n-1$, $\nu = 1,..., p-1$, $i = 0,..., m-1$ and $\mu = 1,..., q$, as follows:

$$N \in I_{j\nu} \quad \text{if and only if there exists } l \leqslant n-1-j \text{ such that } (N, l) \in I(\nu, 1) \cup \cdots \cup I(\nu, q), \tag{3.1}$$

$$N \in J_{i\mu} \quad \text{if and only if there exists } k \leqslant m-1-i \text{ such that } (k, N) \in I(1, \mu) \cup \cdots \cup I(p, \mu). \tag{3.2}$$

Finally, for $j = 0,..., n-1$, let

$$I_{jp} = \{0,..., m-1\}.$$

We consider the one-dimensional spline functions

$$s_j(x) = \sum_{\nu=1}^{p} \sum_{i \in I_{j\nu}} \omega_\nu^{ij} (x - x_\nu)_+^{2m-1-i}, \qquad j = 0,..., n-1,$$

and

$$t_i(y) = \sum_{\mu=1}^{q} \sum_{j \in J_{i\mu}} \rho_\mu^{ij} (y - y_\mu)_+^{2n-1-j}, \qquad i = 0,..., m-1.$$

Furthermore, let

$$f(x, y) = \sum_{\nu=1}^{p} \sum_{\mu=1}^{q} \sum_{(i,j) \in I(\nu,\mu)} \sigma_{\nu\mu}^{ij} (x - x_\nu)_+^{2m-1-i} (y - y_\mu)_+^{2n-1-j}$$

and denote by

$$\Omega_{mn} = \Omega_{mn}(I(\nu, \mu), x_\nu, y_\mu, p, q)$$

the set of all functions $S(x, y)$ which can be written in the form

$$S(x, y) = f(x, y) + \sum_{j=0}^{n-1} y^j s_j(x) + \sum_{i=0}^{m-1} x^i t_i(y) + P(x, y),$$

where $P(x, y) \in \Pi_{m-1,n-1}$ and

$$\frac{\partial^m}{\partial x^m} f(x, y) \equiv 0 \quad \text{for} \quad x \geqslant x_p,$$

$$\frac{\partial^n}{\partial y^n} f(x, y) \equiv 0 \quad \text{for} \quad y \geqslant y_q,$$

$$\frac{d^m}{dx^m} s_j(x) \equiv 0 \quad \text{for} \quad x \geqslant x_p, \quad j = 0,..., n-1,$$

$$\frac{d^n}{dy^n} t_i(y) \equiv 0 \quad \text{for} \quad y \geqslant y_q, \quad i = 0,..., m-1.$$

As in Ref. [4] we call any $S \in \Omega_{mn}$ a two-dimensional natural g spline for the knots (x_ν, y_μ), $\nu = 1,..., p$, $\mu = 1,..., q$, the sets $I(\nu, \mu)$ and order (m, n).

Let $\beta_{\nu\mu}^{ij}$ be arbitrary real numbers and consider the following interpolation problem: Find $S \in \Omega_{mn}$ satisfying

$$\frac{\partial^{i+j}}{\partial x^i \, \partial y^j} S(x_\nu, y_\mu) = \beta_{\nu\mu}^{ij}, \qquad (i, j) \in I(\nu, \mu),$$
$$\nu = 1,..., p, \quad \mu = 1,..., q. \tag{3.3}$$

We say the interpolation problem is (m, n)-poised [4], provided that

$$\frac{\partial^{i+j}}{\partial x^i \, \partial y^j} S(x_\nu, y_\mu) = 0, \qquad (i, j) \in I(\nu, \mu),$$
$$\nu = 1,..., p, \quad \mu = 1,..., q,$$

imply $S(x, y) \equiv 0$ for all $S \in \Omega_{mn}$ such that $f(x, y) \equiv 0$.

In Ref. [4] it was shown that if the interpolation problem is (m, n)-poised, then there exists a unique $S_0 \in \Omega_{mn}$ which satisfies (3.3). The parameters determining this $S_0(x, y)$ can be obtained by solving a system of linear equations.

Now suppose the interpolation problem is (m, n)-poised. For $(i, j) \in I(\nu, \mu)$, $\nu = 1,..., p$, $\mu = 1,..., q$, let $S_{\nu\mu}^{ij}(x)$ denote the uniquely determined element of Ω_{mn} which satisfies the relations

$$\frac{\partial^{k+l}}{\partial x^k \, \partial y^l} S_{\nu\mu}^{ij}(x_\xi, y_\eta) = \begin{cases} 1 & \text{if} \quad k = i, \quad l = j, \quad \xi = \nu, \quad \eta = \mu, \\ 0 & \text{if} \quad k \neq i, \quad l \neq j, \quad \xi \neq \nu, \quad \text{or } \eta \neq \mu, \end{cases}$$
$$(k, l) \in I(\xi, \eta), \quad \xi = 1,..., p, \quad \eta = 1,..., q. \tag{3.4}$$

For functions $g(x, y)$ with appropriate differentiability properties we consider the interpolation formula

$$g(x, y) = \sum_{\nu=1}^{p} \sum_{\mu=1}^{q} \sum_{(i,j) \in I(\nu,\mu)} \frac{\partial^{i+j}}{\partial x^i \, \partial y^j} g(x_\nu, y_\mu) \, S_{\nu\mu}^{ij}(x, y) + \mathscr{R}(g). \tag{3.5}$$

The sum on the right side represents the element of Ω_{mn} which satisfies (3.3) with

$$\beta_{\nu\mu}^{ij} = \frac{\partial^{i+j}}{\partial x^i \, \partial y^j} g(x_\nu, y_\mu).$$

Following Schoenberg's terminology in the one-dimensional case [7], we call (3.5) the (two-dimensional) g-spline interpolation formula of order (m, n). It is exact for all elements of Ω_{mn}.

In the next section we shall assume that for $\nu = 1,..., p$ and $\mu = 1,..., q$, the sets $I(\nu, q)$ and $I(p, \mu)$ have the following properties:

If for any $i_0 \in \{0,..., m-1\}$ there exists some $j_0 \in \{0,..., n-1\}$ and some $\mu_0 \in \{1,..., q\}$ such that $(i_0, j_0) \in I(\nu, \mu_0)$, then $(i_0, j) \in I(\nu, q)$ for all $j \in \{0,..., n-1\}$. (3.6)

If for any $j_0 \in \{0,..., n-1\}$ there exists some $i_0 \in \{0,..., m-1\}$ and some $\nu_0 \in \{1,..., p\}$ such that $(i_0, j_0) \in I(\nu_0, \mu)$, then $(i, j_0) \in I(p, \mu)$ for all $i \in \{0,..., m-1\}$. (3.7)

It is easily seen that under these assumptions

$$I_{0\nu} = I_{1\nu} = \cdots = I_{n-1,\nu}, \qquad \nu = 1,..., p,$$

and

$$J_{0\mu} = J_{1\mu} = \cdots = J_{m-1,\mu}, \qquad \mu = 1,..., q.$$

As a simple example of an interpolation problem (3.3) which is (m, n)-poised and has the properties (3.6) and (3.7), we consider the case $m = n = 2$ and $p, q > 2$. Furthermore, we assume that the values of $S(x, y)$ at the mesh points (x_ν, y_μ), the normal derivates of $S(x, y)$ at the boundary points of the mesh, and the cross derivative at the four corners of the mesh are prescribed. That is, we assume

$$\begin{aligned}
&I(1, 1) = I(1, q) = I(p, 1) = I(p, q) = \{(0, 0), (0, 1), (1, 0), (1, 1)\}, \\
&I(1, \mu) = I(p, \mu) = \{(0, 0), (1, 0)\}, \qquad \mu = 2,..., q-1, \\
&I(\nu, 1) = I(\nu, q) = \{(0, 0), (0, 1)\}, \qquad \nu = 2,..., p-1, \\
&I(\nu, \mu) = \{(0, 0)\}, \qquad \nu = 2,..., p-1, \qquad \mu = 2,..., q-1.
\end{aligned}$$

Under these assumptions,

$$\begin{aligned}
I_{j1} = I_{jp} = \{0, 1\}, &\qquad j = 0, 1, \\
I_{j\nu} = \{0\}, \qquad j = 0, 1, &\qquad \nu = 2,..., p-1,
\end{aligned}$$

and

$$J_{i1} = J_{iq} = \{0, 1\}, \qquad i = 0, 1,$$
$$J_{i\mu} = \{0\}, \qquad i = 0, 1, \qquad \mu = 2,..., q - 1.$$

Therefore, as is not difficult to verify, the interpolation problem (3.3) is (m, n)-poised. Finally, the definitions of $I(\nu, q)$ and $I(p, \mu)$ imply immediately that (3.6) and (3.7) are fulfilled.

4. The Best Approximation Formula

Having the spline interpolation formula (3.5) at our disposal we can now prove the main theorem of this paper. It states that the best approximation (2.2) to $\ell(g)$ is obtained by applying ℓ to both sides of the two-dimensional g-spline interpolation formula (3.5). In the one-dimensional case this fact was established by Schoenberg [6] and Ahlberg and Nilson [1].

Theorem (4.1). *Suppose that the interpolation problem* (3.3) *is* (m, n)-*poised and satisfies* (3.6) *and* (3.7). *Furthermore, let* $x_p = b_1$ *and* $y_q = b_2$. *Then the coefficients* $c_{\nu\mu}^{ij}$ *minimize the function* (2.5) *under the side conditions* (2.6), *if and only if*

$$c_{\nu\mu}^{ij} = \ell(S_{\nu\mu}^{ij}), \qquad (i, j) \in I(\nu, \mu), \qquad \nu = 1,..., p, \qquad \mu = 1,..., q.$$

Here $S_{\nu\mu}^{ij}(x, y)$ *are the spline functions defined by* (3.4).

Proof. We shall prove this theorem by generalizing a method which was used by Greville [3] and Schoenberg [6] in establishing Theorem (4.1) for the one-dimensional case.

Let ℓ be of type (2.1) and $g \in C^{mn}[R]$. We consider the linear functionals

$$\mathscr{R}^0(g) = \ell(g) - \sum_{\nu=1}^{p} \sum_{\mu=1}^{q} \sum_{(i,j)\in I(\nu,\mu)} c_{\nu\mu}^{ij} \frac{\partial^{i+j}}{\partial x^i \, \partial y^j} g(x_\nu, y_\mu) \tag{1}$$

and

$$\mathscr{R}^1(g) = \ell(g) - \sum_{\nu=1}^{p} \sum_{\mu=1}^{q} \sum_{(i,j)\in I(\nu,\mu)} d_{\nu\mu}^{ij} \frac{\partial^{i+j}}{\partial x^i \, \partial y^j} g(x_\nu, y_\mu), \tag{2}$$

where $c_{\nu\mu}^{ij} = \ell(S_{\nu\mu}^{ij})$ and the coefficients $d_{\nu\mu}^{ij}$ are only required to satisfy the equations (2.3).

Since we know that the spline interpolation formula (3.5) is exact for all

$S \in \Omega_{mn}$ it follows that the approximation formula (1) obtained from (3.5) by operating on both sides with ℓ is also exact for all $S \in \Omega_{mn}$, i.e.,

$$\mathscr{R}^0(S) = 0 \qquad \text{for all} \quad S \in \Omega_{mn}. \tag{3}$$

Hence, also,

$$\mathscr{R}^0(g) = \mathscr{R}^1(g) = 0 \qquad \text{for all} \quad g \in \Pi_{m-1,n-1}. \tag{4}$$

Let

$$M^1(x, y) = \sum_{\nu=1}^{p} \sum_{\mu=1}^{q} \sum_{(i,j)\in I(\nu,\mu)} (-1)^{i+j} \frac{d_{\nu\mu}^{ij} - c_{\nu\mu}^{ij}}{(2m-1-i)!\,(2n-1-j)!} \times (x - x_\nu)_+^{2m-1-i} (y - y_\mu)_+^{2n-1-j};$$

then

$$(\mathscr{R}_{st}^1 - \mathscr{R}_{st}^0)\left(\frac{(x-s)_+^{m-1}(y-t)_+^{n-1}}{(m-1)!\,(n-1)!}\right) = \frac{\partial^{m+n}}{\partial x^m\, \partial y^n} M^1(x, y). \tag{5}$$

Because

$$\begin{aligned}(x-s)_+^{m-1}(y-t)_+^{n-1} &= (-1)^m (s-x)_+^{m-1}(y-t)_+^{n-1} + (-1)^n (x-s)_+^{m-1}(t-y)_+^{n-1} \\ &\quad + (-1)^{m+n-1}(s-x)_+^{m-1}(t-y)_+^{n-1} + (x-s)^{m-1}(y-t)^{n-1},\end{aligned}$$ [1]

it follows from (1), (2), (4), and (5) that

$$\begin{aligned}\frac{\partial^{m+n}}{\partial x^m\,\partial y^n} M^1(x, y) &= \frac{1}{(m-1)!\,(n-1)!}(\mathscr{R}_{st}^1 - \mathscr{R}_{st}^0)((-1)^m (s-x)_+^{m-1}(y-t)_+^{n-1} \\ &\quad + (-1)^n (x-s)_+^{m-1}(t-y)_+^{n-1} + (-1)^{m+n-1}(s-x)_+^{m-1}(t-y)_+^{n-1}) \\ &= \sum_{\nu=1}^{p}\sum_{\mu=1}^{q}\sum_{(i,j)\in I(\nu,\mu)} \frac{d_{\nu\mu}^{ij} - c_{\nu\mu}^{ij}}{(m-1-i)!\,(n-1-j)!}((-1)^{m-j}(x_\nu - x)_+^{m-1-i} \\ &\quad \times (y - y_\mu)_+^{n-1-j} + (-1)^{n-i}(x - x_\nu)_+^{m-1-i}(y_\mu - y)_+^{n-1-j} \\ &\quad + (-1)^{m+n-1-i-j}(x_\nu - x)_+^{m-1-i}(y_\mu - y)_+^{n-1-j}).\end{aligned}$$

[1] If $m = 1$ we define $(s-x)_+^0(y-t)_+^{n-1} = (x-s)_+^0(t-y)_+^{n-1} = (s-x)_+^0(t-y)_+^{n-1} = 0$ for $x = s$, and if $n = 1$ we define $(s-x)_+^{m-1}(y-t)_+^0 = (x-s)_+^{m-1}(t-y)_+^0 = (s-x)_+^{m-1}(t-y)_+^0 = 0$ for $y = t$.

It is a direct consequence of this equality and of the definition of $M^1(x, y)$ that

$$\frac{\partial^{m+n}}{\partial x^m \, \partial y^n} M^1(x, y) \equiv 0 \tag{6}$$

if $x < x_1$, $y < y_1$, or if $x \geqslant x_p$, $y \geqslant y_q$.

Now let

$$s_l(x) = \frac{\partial^{2n-1-l}}{\partial y^{2n-1-l}} M^1(x, b_2 + 0), \qquad l = 0,..., n-1, \tag{7}$$

and

$$t_k(y) = \frac{\partial^{2m-1-k}}{\partial x^{2m-1-k}} M^1(b_1 + 0, y), \qquad k = 0,..., m-1. \tag{8}$$

Then it follows from (6) and the definition of $M^1(x, y)$ that

$$s_l^{(m)}(x) \equiv 0 \quad \text{for} \quad x < x_1 \quad \text{and} \quad x \geqslant x_p, \tag{9}$$

and

$$t_k^{(n)}(y) \equiv 0 \quad \text{for} \quad y < y_1 \quad \text{and} \quad y \geqslant y_q. \tag{10}$$

Furthermore, for every $\xi \in \{1,..., p\}$, the contribution to $s_l(x)$ due to a fixed knot x_ξ is

$$\sum_{\mu=1}^{q} \sum_{\substack{(i,j)\in I(\xi,\mu)\\ j\leqslant l}} (-1)^{i+j} \frac{d_{\xi\mu}^{ij} - c_{\xi\mu}^{ij}}{(2m-1-i)!\,(l-j)!} (b_2 - y_\mu)^{l-j} (x - x_\xi)_+^{2m-1-i}. \tag{11}$$

Similarly, for every $\eta \in \{1,..., q\}$, the contribution to $t_k(y)$ due to a fixed knot y_η is

$$\sum_{\nu=1}^{p} \sum_{\substack{(i,j)\in I(\nu,\eta)\\ i\leqslant k}} (-1)^{i+j} \frac{d_{\nu\eta}^{ij} - c_{\nu\eta}^{ij}}{(k-i)!\,(2n-1-j)!} (b_1 - x_\nu)^{k-i} (y - y_\eta)_+^{2n-1-j}. \tag{12}$$

Thus, by (3.6) and (3.7), $s_l^{(2m-1-i)}(x)$ is continuous at $x = x_\xi$ if $i \notin I_{l\xi}$, and $t_k^{(2n-1-j)}(y)$ is continuous at $y = y_\eta$ if $j \notin J_{k\eta}$, where $I_{l\xi}$, $l = 0,..., n-1$, $\xi = 1,..., p$, and $J_{k\eta}$, $k = 0,..., m-1$, $\eta = 1,..., q$, are defined by (3.1) and (3.2), respectively.

It follows therefore from (9) and (10) that

$$(b_2 - y)^l \, s_l(x) \in \Omega_{mn} \quad \text{for} \quad l = 0,..., n-1, \tag{13}$$

$$(b_1 - x)^k \, t_k(y) \in \Omega_{mn} \quad \text{for} \quad k = 0,..., m-1. \tag{14}$$

Since (3.6) and (3.7) hold, we conclude from (7), (8), (11), and (12) that there are coefficients $\delta_{\nu\mu}^{ij}$, $(i, j) \in I_{\nu\mu}$, $\nu = 1,..., p$, $\mu = 1,..., q$, such that

$$\delta_{\nu\mu}^{ij} = 0 \qquad \text{for} \quad (i, j) \in I_{\nu\mu}, \quad \nu = 1,..., p-1, \quad \mu = 1,..., q-1, \tag{15}$$

satisfying

$$N(x, y) = \sum_{\nu=1}^{p} \sum_{\mu=1}^{q} \sum_{(i,j)\in I(\nu,\mu)} \frac{\delta_{\nu\mu}^{ij}}{(2m-1-i)!\,(2n-1-j)!} \times (x - x_\nu)_+^{2m-1-i} (y - y_\mu)_+^{2n-1-j},$$

$$\frac{\partial^{2n-1-l}}{\partial y^{2n-1-l}} N(x, b_2 + 0) = s_l(x), \qquad l = 0,..., n-1, \tag{16}$$

and

$$\frac{\partial^{2m-1-k}}{\partial x^{2m-1-k}} N(b_1 + 0, y) = t_k(y), \qquad k = 0,..., m-1. \tag{17}$$

Now let

$$M^2(x, y) = M^1(x, y) - N(x, y).$$

Then, by (7), (8), (16), and (17),

$$\frac{\partial^{2n-1-l}}{\partial y^{2n-1-l}} M^2(x, b_2 + 0) \equiv 0, \qquad l = 0,..., n-1,$$

$$\frac{\partial^{2m-1-k}}{\partial x^{2m-1-k}} M^2(b_1 + 0, y) \equiv 0, \qquad k = 0,..., m-1.$$

Using Lemma (2.2) of Ref. [4] we, obtain from these identities that

$$\frac{\partial^m}{\partial x^m} M^2(x, y) \equiv 0 \qquad \text{for} \quad x \geqslant x_p,$$

and

$$\frac{\partial^n}{\partial y^n} M^2(x, y) \equiv 0 \qquad \text{for} \quad y \geqslant y_q.$$

Hence, by the definitions of $M^1(x, y)$ and $N(x, y)$,

$$M^2(x, y) \in \Omega_{mn}. \tag{18}$$

If we put

$$\hat{s}_l(x) = \frac{\partial^l}{\partial y^l} M^2(x, b_2), \qquad l = 0,..., n-1, \tag{19}$$

and

$$\hat{t}_k(y) = \frac{\partial^k}{\partial y^k} M^2(b_1, y), \qquad k = 0,\ldots, m-1, \tag{20}$$

it follows that

$$\hat{s}_l^{(m)}(x) \equiv 0 \qquad \text{for} \quad x \geqslant x_p, \quad l = 0,\ldots, n-1,$$

$$\hat{t}_k^{(n)}(y) \equiv 0 \qquad \text{for} \quad y \geqslant y_q, \quad k = 0,\ldots, m-1.$$

Since it is an immediate consequence of (3.6) and (3.7) that $\hat{s}_l^{(2m-1-i)}(x)$ is continuous at $x = x_\nu$, $\nu = 1,\ldots, p$, if $i \notin I_{l\nu}$, and $\hat{t}_k^{(2n-1-j)}(y)$ is continuous at $y = y_\mu$, $\mu = 1,\ldots, q$, if $j \notin J_{k\mu}$, we have

$$\frac{(b_2 - y)^l}{l!} \hat{s}_l(x) = \frac{(b_2 - y)^l}{l!} \sum_{\nu=1}^{p} \sum_{i \in I_{l\nu}} \omega_\nu^{ij}(x - x_\nu)_+^{2m-1-i} \in \Omega_{mn},$$

$$l = 0,\ldots, n-1, \tag{21}$$

$$\frac{(b_1 - x)^k}{k!} \hat{t}_k(y) = \frac{(b_1 - x)^k}{k!} \sum_{\mu=1}^{q} \sum_{j \in J_{i\mu}} \rho_\mu^{ij}(y - y_\mu)_+^{2n-1-j} \in \Omega_{mn},$$

$$k = 0,\ldots, m-1. \tag{22}$$

For $i = 0, 1$, let

$$K_{1l}^i(x) = \frac{1}{(m-1)!\, l!} \mathscr{R}_{st}^i((x - s)_+^{m-1} (b_2 - t)^l), \qquad l = 0,\ldots, n-1, \tag{23}$$

$$K_{2k}^i(y) = \frac{1}{k!\,(n-1)!} \mathscr{R}_{st}^i((b_1 - s)^k (y - t)_+^{n-1}), \qquad k = 0,\ldots, m-1, \tag{24}$$

$$K^i(x, y) = \frac{1}{(m-1)!\,(n-1)!} \mathscr{R}_{st}^i((x - s)_+^{m-1} (y - t)_+^{n-1}). \tag{25}$$

It follows from (3), (4), (19)–(24), Theorem (2.1), and Remark 1 that

$$0 = \frac{1}{l!} \mathscr{R}^0((b_2 - y)^l \hat{s}_l(x)) = (-1)^{m+1} \int_{a_1}^{b_1} K_{1l}^0(x)\, \hat{s}_l^{(m)}(x)\, dx$$

$$= (-1)^{m+l} \int_{a_1}^{b_1} K_{1l}^0(x) \frac{\partial^{m+1}}{\partial x^m\, \partial y^l} M^2(x, b_2)\, dx, \qquad l = 0,\ldots, n-1, \tag{26}$$

$$0 = \frac{1}{k!} \mathscr{R}^0((b_1 - x)^k \hat{t}_k(y)) = (-1)^{n+k} \int_{a_2}^{b_2} K_{2k}^0(y)\, \hat{t}_k^{(n)}(y)\, dy$$

$$= (-1)^{n+k} \int_{a_2}^{b_2} K_{2k}^0(y) \frac{\partial^{k+n}}{\partial x^k\, \partial y^n} M^2(b_1, y)\, dy, \qquad k = 0,\ldots, m-1. \tag{27}$$

Since by (3) and (18), $\mathscr{R}^0(M^2(x, y)) = 0$, we obtain from (4), (25)–(27), Theorem (2.1), and Remark 1 that

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} K^0(x, y) \frac{\partial^{m+n}}{\partial x^m \, \partial y^n} M^2(x, y) \, dy \, dx = 0.$$

Observe that, by (15), $N(x, y) \equiv 0$ for $x < x_p = b_1$ and $y < y_q = b_2$. Hence, the above equality implies

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} K^0(x, y) \frac{\partial^{m+n}}{\partial x^m \, \partial y^n} M^1(x, y) \, dy \, dx = 0,$$

which, by (5) and (25), is equivalent to

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} K^0(x, y)(K^1(x, y) - K^0(x, y)) \, dy \, dx = 0.$$

An immediate consequence of the last equality is

$$\int_{a_1}^{b_1} \int_{a_2}^{b_2} (K^1(x, y))^2 \, dy \, dx$$

$$= \int_{a_1}^{b_1} \int_{a_2}^{b_2} (K^0(x, y))^2 \, dy \, dx + \int_{a_1}^{b_1} \int_{a_2}^{b_2} (K^1(x, y) - K^0(x, y))^2 \, dy \, dx. \quad (28)$$

Furthermore, by (3), (13), and (14),

$$\mathscr{R}^0((b_2 - y)^l \, s_l(x)) = \mathscr{R}^0((b_1 - x)^k \, t_k(y)) = 0,$$
$$l = 0, \dots, n - 1, \quad k = 0, \dots, m - 1.$$

Thus, it follows from (4), (23), (24), Theorem (2.1), and Remark 1 that

$$\int_{a_1}^{b_1} K_{1l}^0(x) \, s_l^{(m)}(x) \, dx = 0, \qquad l = 0, \dots, n - 1, \quad (29)$$

and

$$\int_{a_2}^{b_2} K_{2k}^0(y) \, t_k^{(n)}(y) \, dy = 0, \qquad k = 0, \dots, m - 1. \quad (30)$$

By (1), (2), (23), (24), and the definition of $M^1(x, y)$,

$$K_{1l}^1(x) - K_{1l}^0(x) = \frac{\partial^{m+2n-1-l}}{\partial x^m \, \partial y^{2n-1-l}} M^1(x, b_2 + 0), \qquad l = 0, \dots, n - 1, \quad (31)$$

$$K_{2k}^1(y) - K_{2k}^0(y) = \frac{\partial^{2m-1-k+n}}{\partial x^{2m-1-k} \, \partial y^n} M^1(b_1 + 0, y), \qquad k = 0, \dots, m - 1. \quad (32)$$

Hence (7), (8), (29), and (30) imply

$$\int_{a_1}^{b_1} K_{1l}^0(x)(K_{1l}^1(x) - K_{1l}^0(x))\,dx = 0, \qquad l = 0,..., n-1,$$

$$\int_{a_2}^{b_2} K_{2k}^0(y)(K_{2k}^1(y) - K_{2k}^0(y))\,dy = 0, \qquad k = 0,..., m-1.$$

From these equalities it follows that

$$\int_{a_1}^{b_1} (K_{1l}^1(x))^2\,dx$$

$$= \int_{a_1}^{b_1} (K_{1l}^0(x))^2\,dx + \int_{a_2}^{b_2} (K_{1l}^1(x) - K_{1l}^0(x))^2\,dx, \qquad l = 0,..., n-1, \quad (33)$$

$$\int_{a_2}^{b_2} (K_{2k}^1(y))^2\,dy$$

$$= \int_{a_2}^{b_2} (K_{2k}^0(y))^2\,dy + \int_{a_2}^{b_2} (K_{2k}^1(y) - K_{2k}^0(y))^2\,dy, \quad k = 0,..., m-1. \quad (34)$$

Hence, (28), (33), and (34) imply that

$$\sum_{l=0}^{n-1} \int_{a_1}^{b_1} (K_{1l}^1(x))^2\,dx + \sum_{k=0}^{m-1} \int_{a_2}^{b_2} (K_{2k}^1(y))^2\,dy + \int_{a_1}^{b_1}\int_{a_2}^{b_2} (K^1(x, y))^2\,dy\,dx$$

$$> \sum_{l=0}^{n-1} \int_{a_1}^{b_1} (K_{1l}^0(x))^2\,dx + \sum_{k=0}^{m-1} \int_{a_2}^{b_2} (K_{2k}^0(y))^2\,dy + \int_{a_1}^{b_1}\int_{a_2}^{b_2} (K^0(x, y))^2\,dy\,dx$$

unless

$$K^1(x, y) - K^0(x, y) \equiv 0,$$
$$K_{1l}^1(x) - K_{1l}^0(x) \equiv 0, \qquad l = 0,..., n-1,$$

and

$$K_{2k}^1(y) - K_{2k}^0(y) \equiv 0, \qquad k = 0,..., m-1,$$

which, by (25), (5), (31), and (32) is equivalent to

$$c_{\nu\mu}^{ij} = d_{\nu\mu}^{ij}, \qquad (i, j) \in I(\nu, \mu), \qquad \nu = 1,..., p, \qquad \mu = 1,..., q.$$

This completes the proof of the theorem.

Remark 2. Let $g \in C^{mn}[R]$ and $(\bar{x}, \bar{y}) \in R$. The linear functional ℓ defined by $\ell(g) = g(\bar{x}, \bar{y})$ is of type (2.1). Hence, it follows from the above theorem that the spline interpolation formula (3.5) is a best interpolation formula in the sense of Sard [5], provided the interpolation problem (3.3) is (m, n)-poised, and (3.6) and (3.7) hold and $x_p = b_1$, $y_q = b_2$.

References

1. J. H. Ahlberg and E. N. Nilson, The approximation of linear functionals, *SIAM. J Numer. Anal.* **3** (1966), 173–182.
2. J. P. Davis, "Interpolation and Approximation," Blaisdell Publ. Co., New York, 1965.
3. T. N. E. Greville, Spline functions, interpolation and numerical quadrature, *in* "Mathematical Methods for Digital Computers" (A. Ralston and H. S. Wilf, Eds.), Vol. 2, John Wiley and Sons, New York, 1967.
4. K. Ritter, Two-dimensional splines and their extremal properties, *ZAMM* **49** (1969), 597–608.
5. A. Sard, "Linear Approximation," American Mathematical Society, Providence, 1963.
6. I. J. Schoenberg, On best approximation of linear operators, *Indag. Math.* **26** (1964), 155–163.
7. I. J. Schoenberg, On the Ahlberg–Nilson extension of spline interpolation: the *g*-splines and their optimal properties, *J. Math. Anal. Appl.* **21** (1968), 207–231.